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ZZ-branes of $N = 2$ super-Liouville theory

Changrim Ahn, Marian Stanishkov* and Masayoshi Yamamoto

Department of Physics, Ewha Womans University
Seoul 120-750, Korea
E-mail: ahn@ewha.ac.kr, stanishkov@dante.ewha.ac.kr, yamamoto@dante.ewha.ac.kr

Abstract: We study conformal boundary conditions and corresponding one-point functions of the $N = 2$ super-Liouville theory using both conformal and modular bootstrap methods. We have found both continuous ('FZZT-branes') and discrete ('ZZ-branes') boundary conditions. In particular, we identify two different types of the discrete ZZ-brane solutions, which are associated with degenerate fields of the $N = 2$ super-Liouville theory.

Keywords: Field Theories in Lower Dimensions, D-branes, Conformal Field Models in String Theory, Boundary Quantum Field Theory.

*On leave of absence from Institute of Nuclear Research and Nuclear Energy, Sofia, Bulgaria.

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1. Introduction

Two-dimensional Liouville field theory (LFT) has been studied for its relevance with non-critical string theories and two-dimensional quantum gravity [1, 2]. Recently, string theory with 2D euclidean black hole geometry [3] has been claimed to be T-dual to the sine-

Liouville theory [4, 5]. A similar duality has been discovered for the $N = 2$ supersymmetric Liouville field theory (SLFT) and a fermionic 2D black hole which is identified with super $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset conformal field theory (CFT) [6, 7].

The LFT and SLFTs are irrational CFTs which have continuously infinite number of primary fields. The irrationality requires new formalisms for exact computation of correlation functions. One very efficient formalism is the conformal bootstrap method which has been first applied to the LFT [8, 9], and later to the $N = 1$ SLFT [10, 11], and the $N = 2$ SLFT [12].

Conformally invariant boundary conditions (BCs) for these models have been also actively studied. The conformal bootstrap method has been extended to the LFT with boundary in [16, 17], and the $N = 1$ SLFT [18, 19]. Recently, the boundary conformal bootstrap of the $N = 2$ SLFT has been studied in [22]. While the LFT and $N = 1$ SLFT are invariant under a dual transformation of the coupling constant $b \to 1/b$, the $N = 2$ SLFT does not have this self-duality. To complement this, a $N = 2$ supersymmetric theory dual to the $N = 2$ SLFT was proposed along with various consistency checks in [12] and proved in [13]. This dual theory has provided additional functional equation which, along with the equation based on the $N = 2$ SLFT, can produce exact correlation functions such as the reflection amplitudes. However, this approach can not be extended to the boundary conformal bootstrap method since the boundary action for the dual $N = 2$ theory is not easy to write down. Due to this deficiency, the boundary conformal bootstrap equations considered in [22] contained an undetermined coefficient and could be used only to check the consistency of the one-point functions obtained by the 'modular bootstrap' method. Moreover, these equations were applicable to the 'neutral' fields due to technical difficulties. One of our motivations in this paper is to derive more functional equations based on the conformal bootstrap method and determine the one-point functions exactly including those of 'non-neutral' fields.

The modular bootstrap method is a generalization of the Cardy formulation for the conformal BCs to the irrational CFTs. This method has been initiated in [17] and extended to the $N = 1$ SLFT [18, 19]. In these works, one-point function of a certain primary field under a specific BC is associated with the boundary amplitude which is the scalar product of the corresponding Ishibashi state and the conformal BC state. The boundary amplitudes satisfy the Cardy conditions which are expressed in terms of the modular S-matrix elements. While this method is proven to be effective, it has been mainly used to check the consistency of the one-point functions derived by the conformal bootstrap method because there are some ambiguities in deriving the boundary amplitudes from the Cardy conditions.

For the $N = 2$ SLFT, the situation becomes different. Since the conformal bootstrap could not be completed, the modular bootstrap method remains the only available formalism to derive the boundary amplitudes for the $N = 2$ SLFT and its T-dual, $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ super-coset CFT in [15] and [22]–[26]. Complete one-point functions of the 'continuous' BC, sometimes called 'FZZT-brane', and for the vacuum BC have been derived, independently, in [23] and [22] using the modular bootstrap method. The results in [23] which are based on rational values of b^2, the coupling constant of the $N = 2$ SLFT, while those in [22]

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are applicable to generic irrational values of b^2. It can be shown that the former results in an 'irrational' limit match with the latter. Meanwhile, the FZZT solutions contain a continuous boundary parameter which describes the geometry of D1-branes. In principle, this parameter must be related to the boundary $N = 2$ SLFT. However, the modular bootstrap alone can not give this relation. In [22] conformal bootstrap equations, although they are not complete enough to determine the one-point functions uniquely, has been used to obtain this relation.

To find all the consistent conformal BCs of the $N = 2$ SLFT and its dual super-coset model is important since they describe the D-branes moving in the black hole background. In addition to the FZZT and the vacuum BCs, there are infinite number of discrete BCs, which are called 'ZZ-branes' in general . For the LFT [17] and $N = 1$ SLFT [18, 19], the ZZ-branes have been constructed in the background geometry of the classical Lobachevskiy plane or the pseudosphere [21]. Our main result in this paper is to construct the general ZZ-brane BCs of the $N = 2$ SLFT.

In section 2 we introduce the $N = 2$ SLFT and its operator contents, in particular, degenerate fields and their properties. Based on this, we derive the functional equations for the one-point functions of primary fields for the continuous BC (FZZT) in section 3. In section 4, we find the ZZ-brane solutions from the functional equations defined on the pseudosphere and discuss their implications for the $N = 2$ SLFT. Modular bootstrap computations for the degenerate fields of the $N = 2$ SLFT are performed in section 5 which provides some consistency checks for the solutions. We conclude this paper with some remarks in section 6.

2. $N = 2$ super-Liouville theory

In this section, we introduce the action and the $N = 2$ superconformal algebra and the primary fields. In particular we discuss the degenerate fields in detail.

2.1 $N = 2$ superconformal algebra

The action of the $N = 2$ SLFT is given by

$$S = \int d^2z \Big[\frac{1}{2\pi} \left(\partial\phi^- \bar{\partial}\phi^+ + \partial\phi^+ \bar{\partial}\phi^- + \psi^- \bar{\partial}\psi^+ + \psi^+ \bar{\partial}\psi^- + \bar{\psi}^- \partial\bar{\psi}^+ + \bar{\psi}^+ \partial\bar{\psi}^- \right) + \\ + i\mu b^2 \psi^- \bar{\psi}^- e^{b\phi^+} + i\mu b^2 \psi^+ \bar{\psi}^+ e^{b\phi^-} + \pi\mu^2 b^2 e^{b(\phi^+ + \phi^-)} \Big] . \tag{2.1}$$

This theory needs a background charge $1/b$ for conformal invariance so that the interaction terms in eq. (2.1) become the screening operators of the CFT.

The stress tensor T, the supercurrents $G^{\pm}$ and the U(1) current J of this CFT are given by

$$T = -\partial\phi^- \partial\phi^+ - \frac{1}{2}(\psi^- \partial\psi^+ + \psi^+ \partial\psi^-) + \frac{1}{2b}(\partial^2\phi^+ + \partial^2\phi^-) , \tag{2.2}$$

$$G^{\pm} = \sqrt{2} i (\psi^{\pm} \partial\phi^{\pm} - \frac{1}{b} \partial\psi^{\pm}), \qquad J = -\psi^- \psi^+ + \frac{1}{b}(\partial\phi^+ - \partial\phi^-) . \tag{2.3}$$

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The $N=2$ super-Virasoro algebra is expressed by the modes of these currents as follows:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n}\,, \\
[L_m, G_r^{\pm}] &= \left(\frac{m}{2}-r\right)G^{\pm}_{m+r}\,, \qquad [J_n, G_r^{\pm}] = \pm G^{\pm}_{n+r}\,, \\
\{G_r^+, G_s^-\} &= 2L_{r+s} + (r-s)J_{r+s} + \frac{c}{3}\left(r^2-\frac{1}{4}\right)\delta_{r+s}\,, \qquad \{G_r^{\pm}, G_s^{\pm}\} = 0\,, \\
[L_m, J_n] &= -nJ_{m+n}\,, \qquad [J_m, J_n] = \frac{c}{3}m\delta_{m+n}\,,
\end{aligned}$$

with the central charge

$$c = 3 + \frac{6}{b^2}\,. \tag{2.4}$$

Super-CFTs have both the Neveu-Schwarz (NS) sector with half-integer fermionic modes and the Ramond (R) sector with integer modes. The primary fields of the $N=2$ SLFT are also classified into the (NS) and the (R) sectors and can be written as follows [28]:

$$N_{\alpha\overline{\alpha}} = e^{\alpha\phi^+ + \overline{\alpha}\phi^-}\,, \qquad R^{(\pm)}_{\alpha\overline{\alpha}} = \sigma^{\pm}e^{\alpha\phi^+ + \overline{\alpha}\phi^-}\,, \tag{2.5}$$

where $\sigma^{\pm}$ are the spin operators.

The conformal dimensions and the U(1) charges of the primary fields $N_{\alpha\overline{\alpha}}$ and $R^{(\pm)}_{\alpha\overline{\alpha}}$ can be obtained:

$$\Delta^{\rm NS}_{\alpha\overline{\alpha}} = -\alpha\overline{\alpha} + \frac{1}{2b}(\alpha+\overline{\alpha})\,, \qquad \Delta^{R}_{\alpha\overline{\alpha}} = \Delta^{\rm NS}_{\alpha\overline{\alpha}} + \frac{1}{8}\,, \tag{2.6}$$

and

$$\omega = \frac{1}{b}(\alpha-\overline{\alpha})\,, \qquad \omega^{\pm} = \omega \pm \frac{1}{2}\,. \tag{2.7}$$

It is more convenient sometimes to use a 'momentum' defined by

$$\alpha + \overline{\alpha} = \frac{1}{b} + 2iP\,, \tag{2.8}$$

and the U(1) charge ω instead of $\alpha, \overline{\alpha}$. In terms of these, the conformal dimension is given by

$$\Delta^{\rm NS} = \frac{1}{4b^2} + P^2 + \frac{b^2\omega^2}{4}\,. \tag{2.9}$$

We will denote the primary fields by $N_{[P,\omega]}$ and $R^{(\pm)}_{[P,\omega]}$ in section 5.

2.2 Degenerate fields

Among the primary fields there is a series of degenerate fields of the $N=2$ SLFT. These degenerate fields can be constructed in the same way as the $N=2$ minimal CFTs in [28]. In this paper we divide these fields into three classes. Class-I degenerate fields are given by

$$N^{\omega}_{m,n} = N_{\alpha^{\omega}_{m,n},\overline{\alpha}^{\omega}_{m,n}}\,, \qquad R^{(\epsilon)\omega}_{m,n} = R^{(\epsilon)}_{\alpha^{\omega}_{m,n},\overline{\alpha}^{\omega}_{m,n}}\,, \tag{2.10}$$

$$\alpha^{\omega}_{m,n} = \frac{1-m+\omega b^2}{2b} - \frac{nb}{2}\,, \qquad \overline{\alpha}^{\omega}_{m,n} = \frac{1-m-\omega b^2}{2b} - \frac{nb}{2}, \quad m,n\in\mathbb{Z}_+\,. \tag{2.11}$$

$N^\omega_{m,n}$ and $R^{(\epsilon)\omega}_{m,n}$ are degenerate at the level mn where the corresponding null states are turned out to be

$$N^\omega_{m,-n}\,, \qquad \text{and} \qquad R^{(\epsilon)\omega}_{m,-n}\,. \tag{2.12}$$

As an example, consider the most simple case $N^\omega_{1,1}$ with the conformal dimension $b^2(\omega^2-1)/4-1/2$ and U(1) charge ω. After simple calculation, one can check that

$$\left[\frac{b^2}{2}(1-\omega^2)J_{-1}+G^+_{-1/2}G^-_{-1/2}-(1-\omega)L_{-1}\right]|N^\omega_{1,1}\rangle \tag{2.13}$$

is annihilated by all the positive modes of the $N=2$ super CFT. Since this state has the U(1) charge ω and dimension $+1$ more than that of $N^\omega_{1,1}$, it corresponds to $|N^\omega_{1,-1}\rangle$ up to a normalization constant. One can continue this analysis to higher values of $m,n>1$ to confirm the statement of eq. (2.12). Notice that the null state structure changes dramatically for $\omega=\pm n$ case. The field $N^{\pm n}_{m,n}$ has a null state $N^{\pm n}_{m,-n}$ at level mn. This $N^{\pm n}_{m,-n}$ field is in fact a class-II degenerate field which we will explain next and has infinite number of null states. Therefore, we exclude the case of $\omega=\pm n$ from class-I fields.

The second class of degenerate fields is denoted by N^ω_m and $R^{(\epsilon)\omega}_m$ and comes in two subclasses, namely, class-IIA and class-IIB. These are given by

$$\text{Class}-\text{IIA}: \qquad N^\omega_m = N_{\alpha^\omega_m,\overline{\alpha}^0_m} \qquad R^{(+)\omega}_m = R^{(+)}_{\alpha^\omega_m,\overline{\alpha}^0_m}\,, \quad \omega>0 \tag{2.14}$$

$$\text{Class}-\text{IIB}: \qquad \tilde{N}^\omega_m = N_{\alpha^0_m,\overline{\alpha}^\omega_m} \qquad R^{(-)\omega}_m = R^{(-)}_{\alpha^0_m,\overline{\alpha}^\omega_m}\,, \quad \omega<0\,. \tag{2.15}$$

Here we have defined

$$\alpha^\omega_m \equiv \frac{1-m+2\omega b^2}{2b}\,, \qquad \overline{\alpha}^\omega_m \equiv \frac{1-m-2\omega b^2}{2b} \tag{2.16}$$

with m a positive odd integer for the (NS) sector and even for the (R) sector.

These fields have null states at level $m/2$ which can be expressed again by eq. (2.14) with ω shifted by $+1$ for class-IIA and by eq. (2.15) with ω shifted by -1 for class-IIB. For $m=1$, these fields become either chiral or anti-chiral field which are annihilated by $G^\pm_{-1/2}$, respectively. For $m=3$, one can construct a linear combination of descendants

$$\left[\left(\omega-\frac{2}{b^2}+1\right)G^+_{-3/2}-G^+_{-1/2}L_{-1}+G^+_{-1/2}J_{-1}\right]|N^\omega_3\rangle \tag{2.17}$$

which satisfies the null state condition. Since this state has U(1) charge $\omega+1$ and dimension $3/2$ higher than that of N^ω_3, it is straightforward to identify it as $N^{\omega+1}_3$ up to a normalization constant. However, it is not the end of the story in this case. The $N^{\omega+1}_3$ field is again degenerate at level $3/2$ because a linear combination of its descendants, exactly eq. (2.17) with ω shifted by $+1$, satisfies the null state condition. This generates $N^{\omega+2}_3$ and it continues infinitely. This infinite null state structure holds for any odd integer m.

This can be illustrated by semi-infinite sequences,

$$\text{Class}-\text{IIA}: N^\omega_m \to N^{\omega+1}_m \to N^{\omega+2}_m \to \cdots \tag{2.18}$$

$$\text{Class}-\text{IIB}: \tilde{N}^\omega_m \to \tilde{N}^{\omega-1}_m \to \tilde{N}^{\omega-2}_m \to \cdots\,. \tag{2.19}$$

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This works similarly for the (R) sector. For example, the null state of the $m = 2$ (R) field is given by

$$G^{\pm}_{-1}|R_2^{(\pm)\omega}\rangle\,. \tag{2.20}$$

We need to deal with class-II neutral ($\omega = 0$) (NS) fields separately. For example, consider the N_3^0 which has two null states

$$\left[\left(1-\frac{2}{b^2}\right)G^{\pm}_{-3/2}-G^{\pm}_{-1/2}L_{-1}+G^{\pm}_{-1/2}J_{-1}\right]|N_3^0\rangle\,, \tag{2.21}$$

which should be identified with N_3^1 and $\tilde{N}_3^{-1}$, respectively. We will call these neutral (NS) degenerate fields as class-III and denote by

$$\text{Class}-\text{III}: \qquad N_m = N_{\alpha_m^0\overline{\alpha}_m^0}\,. \tag{2.22}$$

The null state structure of the class-III fields has an infinite sequence in both directions,

$$\text{Class}-\text{III}: \qquad \cdots\leftarrow \tilde{N}_m^{-2}\leftarrow \tilde{N}_m^{-1}\leftarrow N_m\rightarrow N_m^1\rightarrow N_m^2\rightarrow\cdots. \tag{2.23}$$

The identity operator is the most simple class-III field with $m = 1$.

The degenerate fields are playing an essential role in both conformal and modular bootstraps. As we will see shortly, some simple degenerate fields satisfy relatively simple operator product expansion (OPE) and make the conformal bootstrap viable. In this paper we will associate the conformal BCs corresponding to the degenerate fields with solutions of the functional equations obtained by the conformal bootstrap.

3. FZZT-Branes

The FZZT-branes can be described as the $N = 2$ SLFT on a half-plane whose BCs are characterized by a continuous parameter. Extending our previous work [22], we complete the conformal bootstrap in this section.

3.1 One-point functions

In this section, we compute exact one-point functions of the (NS) and (R) bulk operators $N_{\alpha\overline{\alpha}}$ and $R^{(+)}_{\alpha\overline{\alpha}}$ of the SLFT with boundary.[1] The boundary preserves the $N = 2$ superconformal symmetry if the following boundary action is added [27]:

$$\begin{aligned}S_{\rm B} = \int_{-\infty}^{\infty}dx\Bigg[&-\frac{i}{4\pi}(\bar{\psi}^+\psi^- + \bar{\psi}^-\psi^+) + \frac{1}{2}a^-\partial_x a^+ - \frac{1}{2}e^{b\phi^+/2}\left(\mu_B a^+ + \frac{\mu b^2}{4\mu_B}a^-\right)\times\\ &\times(\psi^- + \bar{\psi}^-) - \frac{1}{2}e^{b\phi^-/2}\left(\mu_B a^- + \frac{\mu b^2}{4\mu_B}a^+\right)(\psi^+ + \bar{\psi}^+) -\\ &-\frac{2}{b^2}\left(\mu_B^2 + \frac{\mu^2 b^4}{16\mu_B^2}\right)e^{b(\phi^+ + \phi^-)/2}\Bigg]\,.\end{aligned} \tag{3.1}$$

[1] From now on we consider $\epsilon = +$ only since the other case is almost the same.

The one-point functions are defined by

$$\langle N_{\alpha\overline{\alpha}}(\xi,\bar{\xi})\rangle = \frac{\mathrm{U}^{\mathrm{NS}}(\alpha,\overline{\alpha})}{|\xi-\bar{\xi}|^{2\Delta^{\mathrm{NS}}_{\alpha\overline{\alpha}}}}\,, \qquad \text{and} \qquad \langle R^{(+)}_{\alpha\overline{\alpha}}(\xi,\bar{\xi})\rangle = \frac{\mathrm{U}^{R}(\alpha,\overline{\alpha})}{|\xi-\bar{\xi}|^{2\Delta^{R}_{\alpha\overline{\alpha}}}}\,, \tag{3.2}$$

with the conformal dimensions given in eq. (2.6). We will simply refer to the coefficients $\mathrm{U}^{\mathrm{NS}}(\alpha,\overline{\alpha})$ and $\mathrm{U}^{R}(\alpha,\overline{\alpha})$ as the one-point functions.

3.2 Conformal bootstrap of the $N = 2$ SLFT

The conformal bootstrap method starts with a two-point correlation function on the half-plane. By choosing one of the two operators as a simple degenerate field, the OPE relation becomes relatively simple. The bootstrap equations arise from considering two different channels; one is taking the OPE before the fields approach on the boundary and the other channel is taking the degenerate field on the boundary where the boundary screening integral based on the boundary action is considered [16].

In practice, due to technical difficulties, we could consider only a few most simple degenerate fields and their OPEs. Usually, these are enough to fix the one-point functions exactly up to overall constants. Let us first consider a two-point function of a neutral degenerate field $N_{-b/2}$ and a general neutral field N_α:[2]

$$G_\alpha(\xi,\xi') = \langle N_{-b/2}(\xi)N_\alpha(\xi')\rangle\,. \tag{3.3}$$

The product of these two fields are expanded into four fields

$$N_{-b/2}N_\alpha = \Big[N_{\alpha-\frac{b}{2}}\Big] + C_{+-}\Big[\psi^+\bar{\psi}^+N_{\alpha-\frac{b}{2},\alpha+\frac{b}{2}}\Big] + C_{-+}\Big[\psi^-\bar{\psi}^-N_{\alpha+\frac{b}{2},\alpha-\frac{b}{2}}\Big] + C_{--}\Big[N_{\alpha+\frac{b}{2}}\Big]\,. \tag{3.4}$$

Here the bracket $[\ldots]$ means the conformal tower of a given primary field. One can see that the second and third terms in the r.h.s. are the super-partners of the corresponding fields. The structure constants can be computed by screening integrals as follows:

$$\begin{aligned} C_{+-}(\alpha) &= C_{-+}(\alpha) = -\pi\mu\frac{\gamma(\alpha b-\frac{b^2}{2}-1)}{\gamma(-\frac{b^2}{2})\gamma(\alpha b)} \\ C_{--}(\alpha) &= 2^{-2b^2-2}\pi^2\mu^2b^4\gamma(1-\alpha b)\gamma\left(\frac{1}{2}-\frac{b^2}{2}-\alpha b\right)\gamma\left(\alpha b-\frac{1}{2}\right)\gamma\left(\alpha b+\frac{b^2}{2}\right), \end{aligned}$$

with $\gamma(x)\equiv\Gamma(x)/\Gamma(1-x)$.

Using these, we can express the two-point function as

$$G_\alpha(\xi,\xi') = \mathrm{U}^{\mathrm{NS}}\left(\alpha-\frac{b}{2}\right)\mathcal{G}^{\mathrm{NS}}_1(\xi,\xi') + C_{--}(\alpha)\,\mathrm{U}^{\mathrm{NS}}\left(\alpha+\frac{b}{2}\right)\mathcal{G}^{\mathrm{NS}}_3(\xi,\xi')\,, \tag{3.5}$$

where the one-point functions of the super-partners vanish due to the supersymmetric boundary. The $\mathcal{G}^{\mathrm{NS}}_i(\xi,\xi')$'s are expressed in terms of the special conformal blocks

$$\mathcal{G}^{\mathrm{NS}}_i(\xi,\xi') = \frac{|\xi'-\bar{\xi}'|^{2\Delta^{\mathrm{NS}}_\alpha-2\Delta^{\mathrm{NS}}_{-b/2}}}{|\xi-\bar{\xi}'|^{4\Delta^{\mathrm{NS}}_\alpha}}\mathcal{F}^{\mathrm{NS}}_i(\eta)\,, \qquad i=1,2,3$$

[2] We will suppress one of the indices of the fields since $\overline{\alpha}=\alpha$.

with

$$\eta = \frac{(\xi - \xi')(\bar{\xi} - \bar{\xi}')}{(\xi - \bar{\xi}')(\bar{\xi} - \xi')} .$$

These conformal blocks can be determined by Dotsenko-Fateev integrals [20]:

$$\begin{aligned} I_i(\eta) &= N_i \mathcal{F}_i^{\rm NS}(\eta), \\ &= \int\int_{\mathcal{C}_i} dx_1 dx_2 \langle N_{-b/2}(\eta) N_\alpha(0) N_{-b/2}(1) \psi^- e^{b\phi^+(x_1)} \psi^+ e^{b\phi^-(x_2)} N_{1/b-\alpha}(\infty) \rangle \\ &= \eta^{\alpha b}(1-\eta)^{-b^2/2} \int\int dx_1 dx_2 (x_1 x_2)^{-\alpha b} \times \\ &\qquad \times [(x_1 - 1)(x_2 - 1)(x_1 - \eta)(x_2 - \eta)]^{b^2/2} (x_1 - x_2)^{-b^2-1} . \end{aligned} \tag{3.6}$$

The index i denotes the three independent integration contours between the branching points $0, \eta, 1, \infty$. The conformal blocks $\mathcal{F}_i^{\rm NS}(\eta)$ are regular at $\eta = 0$. Since we are interested in the limit $\eta \to 1$, we need to introduce another blocks which are well defined in that limit. This can be provided by $\tilde{I}_i$ which is given by the same integral (3.6) with a different contour as explained in [20]. Using this, we can define another set of conformal blocks as

$$\tilde{I}_i(\eta) = \tilde{N}_i \tilde{\mathcal{F}}_i^{\rm NS}(\eta) \tag{3.7}$$

so that $\tilde{\mathcal{F}}_j^{\rm NS}(\eta)$ are regular at $\eta = 1$. The monodromy relations between the conformal blocks are given by [20]:

$$\mathcal{F}_i^{\rm NS}(\eta) = \sum_{j=1}^{3} \alpha_{ij} \tilde{\mathcal{F}}_j^{\rm NS}(\eta) , \tag{3.8}$$

with

$$\begin{aligned} \alpha_{13} &= \frac{\Gamma(-1)\Gamma(-\frac{1}{2} - \frac{b^2}{2})}{\Gamma(-\frac{b^2}{2})\Gamma(\frac{1}{2})} \frac{\Gamma(\alpha b - \frac{b^2}{2})\Gamma(\alpha b + \frac{1}{2})}{\Gamma(\alpha b)\Gamma(\alpha b - \frac{b^2}{2} - \frac{1}{2})} \\ \alpha_{23} &= \frac{2\Gamma(-1)\Gamma(-1 - b^2)}{\Gamma(-\frac{b^2}{2})\Gamma(1 + \frac{b^2}{2})} \frac{\Gamma(\alpha b + \frac{b^2}{2} + 1)\Gamma(2 - \alpha b + \frac{b^2}{2})}{\Gamma(\alpha b)\Gamma(-\alpha b + 1)} \\ \alpha_{33} &= \frac{\Gamma(-1)\Gamma(-\frac{1}{2} - \frac{b^2}{2})}{\Gamma(-\frac{b^2}{2})\Gamma(\frac{1}{2})} \frac{\Gamma(\frac{3}{2} - \alpha b)\Gamma(1 - \frac{b^2}{2} - \alpha b)}{\Gamma(1 - \alpha b)\Gamma(\frac{1}{2} - \frac{b^2}{2} - \alpha b)} . \end{aligned}$$

Here we have written only those for $\tilde{\mathcal{F}}_3^{\rm NS}$ because we are interested in the identity operator in the intermediate channel. Notice that this calculation involves a divergent constant $\Gamma(-1)$. We will show that this factor is canceled in the functional equation.

The two-point function $G_\alpha(\xi, \xi')$ in the other channel can be computed as the two fields approach the boundary. When the degenerate field $N_{-b/2}$ approaches, it can be expanded in boundary operators including the boundary identity operator. For the boundary identity operator, a special bulk-boundary structure constant $\mathcal{R}(-b/2)$ can be computed by the boundary action (3.1):

$$\begin{aligned} \mathcal{R}(-b/2) &= -\overline{\mu}_B^2 \int\int dx_1 dx_2 \left\langle N_{-\frac{b}{2}}(i/2) \psi^- e^{b\phi^+(x_1)/2} \psi^+ e^{b\phi^-(x_2)/2} e^{\frac{1}{2b}(\phi^+ + \phi^-)(\infty)} \right\rangle \\ &= -2^{-b^2+1} \sqrt{\pi} \overline{\mu}_B^2 \frac{\Gamma(0)\Gamma(-\frac{1}{2} - \frac{b^2}{2})}{\Gamma(-\frac{b^2}{2})} \end{aligned} \tag{3.9}$$

with

$$\overline{\mu}_B{}^2 = \mu_B^2 + \frac{\mu^2 b^4}{16\mu_B^2}\,. \tag{3.10}$$

This constant also contains a singular factor $\Gamma(0)$. After dividing out these factors on the both sides of the equation, we obtain the following bootstrap equation

$$\begin{aligned} 2^{-b^2+1}\pi\overline{\mu}_B^2\,\mathrm{U}^{\mathrm{NS}}(\alpha) &= \frac{\Gamma(\alpha b - \frac{b^2}{2})\Gamma(\alpha b + \frac{1}{2})}{\Gamma(\alpha b)\Gamma(\alpha b - \frac{b^2}{2} - \frac{1}{2})}\,\mathrm{U}^{\mathrm{NS}}\left(\alpha - \frac{b}{2}\right) + \\ &\quad + 2^{-2-2b^2}\pi^2 b^4 \mu^2 \frac{\Gamma(\alpha b - \frac{1}{2})\Gamma(\alpha b + \frac{b^2}{2})}{\Gamma(\alpha b)\Gamma(\alpha b + \frac{b^2}{2} + \frac{1}{2})}\,\mathrm{U}^{\mathrm{NS}}\left(\alpha + \frac{b}{2}\right). \end{aligned} \tag{3.11}$$

It turns out that a similar functional equation for the (R) field can not be obtained in this way. Instead we will show shortly that the other functional equations can be used to find (R) one-point functions.

3.3 Conformal bootstrap based on the dual action

In [12] it was proposed that the $N = 2$ SLFT is dual to the theory with the action

$$\begin{aligned} S = \int d^2 z \Bigg[&\frac{1}{2\pi}(\partial\phi^-\bar\partial\phi^+ + \partial\phi^+\bar\partial\phi^- + \psi^-\bar\partial\psi^+ + \psi^+\bar\partial\psi^- + \bar\psi^-\partial\bar\psi^+ + \bar\psi^+\partial\bar\psi^-) + \\ &+ \frac{\tilde\mu}{b^2}(\partial\phi^- - \frac{1}{b}\psi^-\psi^+)(\bar\partial\phi^+ - \frac{1}{b}\bar\psi^+\bar\psi^-)e^{\frac{1}{b}(\phi^+ + \phi^-)} \Bigg]\,, \end{aligned} \tag{3.12}$$

where $\tilde\mu$ is the dual cosmological constant. One can derive functional equations for the one-point functions of neutral fields by using the screening operator in the dual action.

We consider the two-point function of the class-II degenerate field $R^+_{-1/2b}$ and a (NS) primary field N_α

$$G_\alpha^{\mathrm{NS}}(\xi,\xi') = \left\langle R^+_{-\frac{1}{2b}}(\xi) N_\alpha(\xi') \right\rangle. \tag{3.13}$$

The OPE of these fields is given by

$$R^+_{-\frac{1}{2b}} N_\alpha = \left[R^+_{\alpha - \frac{1}{2b}}\right] + C^{\mathrm{NS}}(\alpha)\left[R^+_{\alpha + \frac{1}{2b}}\right], \tag{3.14}$$

where the structure constant $C^{\mathrm{NS}}(\alpha)$ was computed in [12] based on the dual action (3.12)

$$C^{\mathrm{NS}}(\alpha) = \pi\tilde\mu\gamma\left(1 + b^{-2}\right)\frac{\Gamma(\frac{2\alpha}{b} - \frac{1}{b^2})\Gamma(1 - \frac{2\alpha}{b})}{\Gamma(1 - \frac{2\alpha}{b} + \frac{1}{b^2})\Gamma(\frac{2\alpha}{b})}\,. \tag{3.15}$$

The two-point function (3.13) can be written as

$$G_\alpha^{\mathrm{NS}}(\xi,\xi') = \frac{|\xi' - \bar\xi'|^{2\Delta_\alpha^{\mathrm{NS}} - 2\Delta^R_{-1/2b}}}{|\xi - \bar\xi'|^{4\Delta_\alpha^{\mathrm{NS}}}}\left[\mathrm{U}^R\left(\alpha - \frac{1}{2b}\right)\mathcal{F}_+^{\mathrm{NS}}(\eta) + C^{\mathrm{NS}}(\alpha)\,\mathrm{U}^R\left(\alpha + \frac{1}{2b}\right)\mathcal{F}_-^{\mathrm{NS}}(\eta)\right], \tag{3.16}$$

where $\mathcal{F}^{\rm NS}_{\pm}(\eta)$ are special conformal blocks. These conformal blocks can be obtained by the following integral

$$\int dx\left\langle R^{+}_{-\frac{1}{2b}}(\eta)N_{\alpha}(0)R^{-}_{-\frac{1}{2b}}(1)\psi^{-}(x)\psi^{+}(x)N_{\frac{1}{b}}(x)N_{\frac{1}{b}-\alpha}(\infty)\right\rangle=$$

$$=\eta^{\frac{\alpha}{b}}(1-\eta)^{-\frac{1}{2b^2}+\frac{3}{4}}\int dx x^{-\frac{2\alpha}{b}}(x-1)^{\frac{1}{b^2}-1}(x-\eta)^{\frac{1}{b^2}-1}. \tag{3.17}$$

Due to two independent contours, this is expressed in terms of the hypergeometric functions

$$\mathcal{F}^{\rm NS}_{+}(\eta)\;=\;\eta^{\frac{\alpha}{b}}(1-\eta)^{-\frac{1}{2b^2}+\frac{3}{4}}F\left(-\frac{1}{b^2}+1,\frac{2\alpha}{b}-\frac{2}{b^2}+1;\frac{2\alpha}{b}-\frac{1}{b^2}+1;\eta\right), \tag{3.18}$$

$$\mathcal{F}^{\rm NS}_{-}(\eta)\;=\;\eta^{-\frac{\alpha}{b}+\frac{1}{b^2}}(1-\eta)^{-\frac{1}{2b^2}+\frac{3}{4}}F\left(-\frac{1}{b^2}+1,-\frac{2\alpha}{b}+1;-\frac{2\alpha}{b}+\frac{1}{b^2}+1;\eta\right). \tag{3.19}$$

In the cross channel, the two-point function can be written as

$$G^{\rm NS}_{\alpha}(\xi,\xi')=\frac{|\xi'-\bar{\xi}'|^{2\Delta^{\rm NS}_{\alpha}-2\Delta^{R}_{-1/2b}}}{|\xi-\bar{\xi}'|^{4\Delta^{\rm NS}_{\alpha}}}\left[B^{\rm NS}_{+}(\alpha)\tilde{\mathcal{F}}^{\rm NS}_{+}(\eta)+B^{\rm NS}_{-}(\alpha)\tilde{\mathcal{F}}^{\rm NS}_{-}(\eta)\right], \tag{3.20}$$

where $B^{\rm NS}_{\pm}(\alpha)$ are the bulk-boundary structure constants and $\tilde{\mathcal{F}}^{\rm NS}_{\pm}(\eta)$ are given by

$$\tilde{\mathcal{F}}^{\rm NS}_{+}(\eta)\;=\;\eta^{\frac{\alpha}{b}}(1-\eta)^{-\frac{1}{2b^2}+\frac{3}{4}}F\left(-\frac{1}{b^2}+1,\frac{2\alpha}{b}-\frac{2}{b^2}+1;-\frac{2}{b^2}+2;1-\eta\right), \tag{3.21}$$

$$\tilde{\mathcal{F}}^{\rm NS}_{-}(\eta)\;=\;\eta^{\frac{\alpha}{b}}(1-\eta)^{\frac{3}{2b^2}-\frac{1}{4}}F\left(\frac{2\alpha}{b},\frac{1}{b^2};\frac{2}{b^2};1-\eta\right). \tag{3.22}$$

The conformal block $\tilde{\mathcal{F}}^{\rm NS}_{-}(\eta)$ corresponds to the boundary identity operator which appears in the boundary as the bulk operator $R^{+}_{-1/2b}$ approaches the boundary. The fusion of N_α to the boundary identity operator is described by the one-point function $\mathrm{U}^{\rm NS}(\alpha)$. On the other hand, the fusion of $R^{+}_{-1/2b}$ is described by a special bulk-boundary structure constant $\mathcal{R}(-1/2b)$ which could be computed as a boundary screening integral with one insertion of the dual boundary interaction if it were known. Therefore, the bulk-boundary structure constant $B^{\rm NS}_{-}(\alpha)$ can be written as $B^{\rm NS}_{-}(\alpha)=\mathcal{R}(-1/2b)\,\mathrm{U}^{\rm NS}(\alpha)$.

Comparing (3.16) with (3.20) and using the monodromy relations between $\mathcal{F}_{\pm}(\eta)$ and $\tilde{\mathcal{F}}_{\pm}(\eta)$, we obtain the following functional equation for the one-point function

$$\mathcal{R}\left(-1/2b\right)\mathrm{U}^{\rm NS}(\alpha)\;=\;\frac{\Gamma(\frac{2\alpha}{b}-\frac{1}{b^2}+1)\Gamma(-\frac{2}{b^2}+1)}{\Gamma(-\frac{1}{b^2}+1)\Gamma(\frac{2\alpha}{b}-\frac{2}{b^2}+1)}\,\mathrm{U}^{R}\left(\alpha-\frac{1}{2b}\right)+$$

$$+\pi\tilde{\mu}\gamma\left(1+b^{-2}\right)\frac{\Gamma(\frac{2\alpha}{b}-\frac{1}{b^2})\Gamma(-\frac{2}{b^2}+1)}{\Gamma(-\frac{1}{b^2}+1)\Gamma(\frac{2\alpha}{b})}\,\mathrm{U}^{R}\left(\alpha+\frac{1}{2b}\right). \tag{3.23}$$

In a similar way, one can derive a functional equation which relates $\mathrm{U}^{R}(\alpha)$ with $\mathrm{U}^{\rm NS}(\alpha\pm 1/2b)$ as is given in [22].

3.4 Conformal bootstrap based on the $N = 2$ minimal CFTs

So far, we have considered only the neutral one-point functions. To consider the charge sector, we need to consider a non-neutral degenerate field. The most simple degenerate field is $N_{-1/b,0}$ which is a special case of class-II field (2.15) with $m = 3$ which has a null state at level 3/2. To use the usual conformal bootstrap screening procedure, we need to consider the dual action proposed in [12] as in previous subsection. However, calculations will be quite complicated. So, we adopt here another strategy which is based on analytic continuation of the $N = 2$ super-minimal CFTs investigated in [28]. These CFTs have central charges $c = 3 - \frac{6}{p+2}$ and their (NS) primary fields are denoted by two integers $l = 0, 1, \ldots$ and $m = -l, -l+2, \ldots, l$. Comparing the central charge of the $N = 2$ SLFT, we can identify $b^2 = -(p+2)$. Furthermore, we can relate the primary fields of the two theories by comparing the conformal dimensions and U(1) charges as follows:

$$N^l_m = e^{-\frac{l+m}{2b}\phi^+ - \frac{l-m}{2b}\phi^-} \tag{3.24}$$

with

$$l = -b(\alpha + \bar{\alpha}) \,, \qquad m = b(\bar{\alpha} - \alpha) \,. \tag{3.25}$$

The degenerate chiral field $N_{-1/b,0}$ is identified with N^1_1. Its OPE with an arbitrary (NS) field is given by

$$N^1_1 N^l_m = N^{l+1}_{m+1} + C^{(NS)}_- N^{l-1}_{m+1} \tag{3.26}$$

and can be translated into that of the $N = 2$ SLFT

$$N_{-1/b,0} N_{\alpha,\bar{\alpha}} = N_{\alpha-1/b,\bar{\alpha}} + C^{(NS)}_- N_{\alpha,\bar{\alpha}+1/b} . \tag{3.27}$$

Indeed, one can check that the structure constants in both cases coincide if eq. (3.25) is imposed. To write a functional equation for $\mathrm{U}(\alpha, \bar{\alpha})$, we follow the same procedure as before. The conformal blocks corresponding to the two terms in eq. (3.27) can be read directly from the $N = 2$ minimal CFT results in [28] using the definition (3.25). The only difference is the normalization constant of the conformal blocks due to the Clebsch-Gordan coefficients in the OPEs of the (NS) fields which was not accounted there. With this normalization the conformal blocks are given by:

$$\frac{1}{1-\alpha b} F\left(\frac{\alpha+\bar{\alpha}}{b} - \frac{2}{b^2}, 1 - \frac{1}{b^2}; \frac{\alpha+\bar{\alpha}}{b} - \frac{1}{b^2} + 1; \eta\right) \tag{3.28}$$

for the first term in the r.h.s. of eq. (3.27) and

$$\frac{1}{\bar{\alpha} b} F\left(-\frac{\alpha+\bar{\alpha}}{b}, 1 - \frac{1}{b^2}; 1 + \frac{1}{b^2} - \frac{\alpha+\bar{\alpha}}{b}; \eta\right) \tag{3.29}$$

for the second one. The structure constant $C^{(NS)}_-$ can also be extracted from [28]:

$$C^{(NS)}_- = -\tilde{\mu}' \frac{(\bar{\alpha} b)^2 \Gamma(-\frac{\alpha+\bar{\alpha}}{b}) \Gamma(\frac{\alpha+\bar{\alpha}}{b} - \frac{1}{b^2})}{\Gamma(1 + \frac{1}{b^2} - \frac{\alpha+\bar{\alpha}}{b}) \Gamma(1 + \frac{\alpha+\bar{\alpha}}{b})} \,, \tag{3.30}$$

where $\tilde{\mu}'$ is given by $\tilde{\mu}$, the cosmological constant of the dual $N=2$ theory,

$$\tilde{\mu}' = 4\pi\tilde{\mu}\gamma(1+b^{-2})b^{-4}\,. \tag{3.31}$$

With all these ingredients we are able to write down new functional equations for the one-point functions of non-neutral primary fields:

$$\begin{aligned}\mathcal{R}(-1/b,0)\,\mathrm{U}^{\mathrm{NS}}(\alpha,\bar{\alpha}) &= \frac{\Gamma(-\frac{2}{b^2})\Gamma(\frac{\alpha+\bar{\alpha}}{b}-\frac{1}{b^2}+1)}{(1-\alpha b)\Gamma(1-\frac{1}{b^2})\Gamma(\frac{\alpha+\bar{\alpha}}{b}-\frac{2}{b^2})}\,\mathrm{U}^{\mathrm{NS}}\left(\alpha-\frac{1}{b},\bar{\alpha}\right)- \\ &\quad -\tilde{\mu}'\frac{(\bar{\alpha}b)\Gamma(-\frac{2}{b^2})\Gamma(\frac{\alpha+\bar{\alpha}}{b}-\frac{1}{b^2})}{\Gamma(1-\frac{1}{b^2})\Gamma(1+\frac{\alpha+\bar{\alpha}}{b})}\,\mathrm{U}^{\mathrm{NS}}\left(\alpha,\bar{\alpha}+\frac{1}{b}\right).\end{aligned} \tag{3.32}$$

Again, we do not know the bulk-boundary structure constant $\mathcal{R}(-1/b,0)$.

The (R) sector can be treated in exactly the same way. Similarly to (3.24), we can identify

$$R^{l}_{m,1} = \sigma^{+}e^{-\frac{l+m+1}{2b}\phi^{+}-\frac{l-m-1}{2b}\phi^{-}}\,. \tag{3.33}$$

The OPE is given by

$$N_{-1/b,0}R^{(+)}_{\alpha,\bar{\alpha}} = R^{(+)}_{\alpha-1/b,\bar{\alpha}} + C^{(R)}_{-}R^{(+)}_{\alpha,\bar{\alpha}+1/b}\,. \tag{3.34}$$

The conformal blocks can be computed as before

$$\frac{1}{3/2-\alpha b}F\left(\frac{\alpha+\bar{\alpha}}{b}-\frac{2}{b^2},1-\frac{1}{b^2};\frac{\alpha+\bar{\alpha}}{b}-\frac{1}{b^2}+1;\eta\right) \tag{3.35}$$

for the first term in the r.h.s. of eq. (3.34) and

$$\frac{1}{\bar{\alpha}b+1/2}F\left(-\frac{\alpha+\bar{\alpha}}{b},1-\frac{1}{b^2};1+\frac{1}{b^2}-\frac{\alpha+\bar{\alpha}}{b};\eta\right) \tag{3.36}$$

for the second one. The structure constant $C^{(R)}_{-}$ is given by

$$C^{(R)}_{-} = -\tilde{\mu}'\frac{(\bar{\alpha}b+\frac{1}{2})^2\Gamma(-\frac{\alpha+\bar{\alpha}}{b})\Gamma(\frac{\alpha+\bar{\alpha}}{b}-\frac{1}{b^2})}{\Gamma(1+\frac{1}{b^2}-\frac{\alpha+\bar{\alpha}}{b})\Gamma(1+\frac{\alpha+\bar{\alpha}}{b})}\,. \tag{3.37}$$

With these, we find the functional equation for the (R) field:

$$\begin{aligned}\mathcal{R}(-1/b,0)\,\mathrm{U}^{R}(\alpha,\bar{\alpha}) &= \frac{\Gamma(-\frac{2}{b^2})\Gamma(\frac{\alpha+\bar{\alpha}}{b}-\frac{1}{b^2}+1)}{(\frac{3}{2}-\alpha b)\Gamma(1-\frac{1}{b^2})\Gamma(\frac{\alpha+\bar{\alpha}}{b}-\frac{2}{b^2})}\,\mathrm{U}^{R}\left(\alpha-\frac{1}{b},\bar{\alpha}\right)- \\ &\quad -\tilde{\mu}'\frac{(\bar{\alpha}b+\frac{1}{2})\Gamma(-\frac{2}{b^2})\Gamma(\frac{\alpha+\bar{\alpha}}{b}-\frac{1}{b^2})}{\Gamma(1-\frac{1}{b^2})\Gamma(1+\frac{\alpha+\bar{\alpha}}{b})}\,\mathrm{U}^{R}\left(\alpha,\bar{\alpha}+\frac{1}{b}\right).\end{aligned} \tag{3.38}$$

One can check that these equations are consistent with the functional equations (3.23) derived in the section 3.3 for neutral fields with $\alpha=\bar{\alpha}$. This justifies the method of analytic continuation used in this subsection. From now on, we will use eqs. (3.32) and (3.38) instead of eq. (3.23) since they are applicable even to non-neutral fields.

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3.5 Solutions

In our previous paper [22], we have derived the one-point functions on the FZZT BC based on the modular bootstrap method and confirmed their validity using a single functional equation. Now it is possible to solve the set of functional equations to determine the one-point functions completely. They are given by

$$\begin{aligned} \mathrm{U}_s^{\mathrm{NS}}(\alpha,\bar\alpha) &= (\pi\mu)^{-\frac{\alpha+\bar\alpha}{b}+\frac{1}{b^2}} \frac{\Gamma\left(1+\frac{\alpha+\bar\alpha}{b}-\frac{1}{b^2}\right)\Gamma\left(b(\alpha+\bar\alpha)-1\right)}{\Gamma\left(\alpha b\right)\Gamma\left(\bar\alpha b\right)} \times \\ &\quad \times \cosh\left[2\pi s\left(\alpha+\bar\alpha-\frac{1}{b}\right)\right], \end{aligned} \tag{3.39}$$

$$\begin{aligned} \mathrm{U}_s^{R}(\alpha,\bar\alpha) &= (\pi\mu)^{-\frac{\alpha+\bar\alpha}{b}+\frac{1}{b^2}} \frac{\Gamma\left(1+\frac{\alpha+\bar\alpha}{b}-\frac{1}{b^2}\right)\Gamma\left(b(\alpha+\bar\alpha)-1\right)}{\Gamma\left(\alpha b-\frac{1}{2}\right)\Gamma\left(\bar\alpha b+\frac{1}{2}\right)} \times \\ &\quad \times \cosh\left[2\pi s\left(\alpha+\bar\alpha-\frac{1}{b}\right)\right]. \end{aligned} \tag{3.40}$$

The continuous parameter s is related nonperturbatively to the boundary cosmological constant μ_B in eq. (3.1) by

$$\overline{\mu}_B{}^2 = \frac{\mu b^2}{2}\cosh(2\pi s b) \tag{3.41}$$

along with the relation between the two cosmological constants

$$4\pi\tilde\mu\gamma(1+b^{-2}) = (\pi\mu)^{2/b^2}\,. \tag{3.42}$$

These results match perfectly with those of the modular bootstrap [22].

4. ZZ-Branes

In this section we are interested in the $N=2$ SLFT on Lobachevskiy plane or pseudosphere which is the geometry of the infinite constant negative curvature surface. Using previous conformal bootstraps, we derive and solve nonlinear functional equations which can provide discrete BCs.

4.1 Pseudosphere geometry

The classical equations of motion for the $N=2$ SLFT can be derived from the lagrangian (2.1):

$$\partial\bar\partial\phi^{\pm} = \pi\mu b^3\left[\pi\mu e^{b(\phi^++\phi^-)} + i\psi^{\pm}\bar\psi^{\pm}e^{b\phi^{\mp}}\right] \tag{4.1}$$

$$\partial\bar\psi^{\pm} = i\pi\mu b^2 e^{b\phi^{\pm}}\psi^{\mp}\,, \qquad \bar\partial\psi^{\pm} = -i\pi\mu b^2 e^{b\phi^{\pm}}\bar\psi^{\mp}\,. \tag{4.2}$$

Assuming that the fermionic fields vanish in the classical limit, we can solve the bosonic fields classically

$$e^{\varphi(z)} = \frac{4R^2}{(1-|z|^2)^2}\,, \tag{4.3}$$

where $\varphi = b(\phi^+ + \phi^-)$ and $R^{-2} = 4\pi^2\mu^2 b^4$. The parameter R is interpreted as the radius of the pseudosphere in which the points at the circle $|z| = 1$ are infinitely far away from any internal point. This circle can be interpreted as the "boundary" of the pseudosphere.

This boundary has a different class of BCs, which are classified by integers whose interpretation is not clear yet. For the $N = 2$ SLFT, we will call the discrete BCs as ZZ-branes following [17] and show that these correspond to the degenerate fields of the $N = 2$ SLFT.

4.2 Conformal bootstrap equations on pseudosphere

In section 3, we have started with two-point correlation functions on a half plane. As the two fields approach on the boundary, the degenerate field is expanded into the boundary operators. On pseudosphere geometry, as they approach on the boundary $\eta \to 1$, the distance between the two points become infinite due to the singular metric. This means that the two-point function is factorized into a product of two one-point functions. For example, the two-point function in eq. (3.3) becomes

$$G_\alpha(\xi, \xi') = \frac{|\xi' - \bar{\xi}'|^{2\Delta_\alpha^{\rm NS} - 2\Delta_{-b/2}^{\rm NS}}}{|\xi - \bar{\xi}'|^{4\Delta_\alpha^{\rm NS}}}\, \mathrm{U}^{\rm NS}(-b/2)\, \mathrm{U}^{\rm NS}(\alpha) \tilde{\mathcal{F}}_3^{\rm NS}(\eta)\,. \tag{4.4}$$

Meanwhile, the computation of the other channel where we take the OPE of the two fields first is identically same as in section 3. Comparing these two results, we can obtain the following nonlinear functional equations for $\mathrm{U}(\alpha)$:

$$\begin{aligned}\mathcal{C}_1\, \mathrm{U}^{\rm NS}(-b/2)\, \mathrm{U}^{\rm NS}(\alpha) &= \frac{\Gamma(\alpha b - \frac{b^2}{2})\Gamma(\alpha b + \frac{1}{2})}{\Gamma(\alpha b)\Gamma(\alpha b - \frac{b^2}{2} - \frac{1}{2})}\, \mathrm{U}^{\rm NS}\left(\alpha - \frac{b}{2}\right) + \\ &\quad + 2^{-2-2b^2}\pi^2 b^4 \mu^2 \frac{\Gamma(\alpha b - \frac{1}{2})\Gamma(\alpha b + \frac{b^2}{2})}{\Gamma(\alpha b)\Gamma(\alpha b + \frac{b^2}{2} + \frac{1}{2})}\, \mathrm{U}^{\rm NS}\left(\alpha + \frac{b}{2}\right)\end{aligned} \tag{4.5}$$

with

$$\mathcal{C}_1 = \frac{\sqrt{\pi}\Gamma(-\frac{b^2}{2})}{\Gamma(-1)\Gamma(-\frac{b^2}{2} - \frac{1}{2})}\,. \tag{4.6}$$

Similarly we can derive functional equations corresponding to eqs. (3.32) and (3.38):

$$\begin{aligned}\mathcal{C}_2 \tilde{U}^{\rm NS}(-1/b, 0)\, \mathrm{U}^{\rm NS}(\alpha, \bar{\alpha}) &= \frac{\Gamma(\frac{\alpha+\bar{\alpha}}{b} - \frac{1}{b^2} + 1)}{(1 - \alpha b)\Gamma(\frac{\alpha+\bar{\alpha}}{b} - \frac{2}{b^2})}\, \mathrm{U}^{\rm NS}\left(\alpha - \frac{1}{b}, \bar{\alpha}\right) - \\ &\quad - \tilde{\mu}' \frac{(\bar{\alpha} b)\Gamma(\frac{\alpha+\bar{\alpha}}{b} - \frac{1}{b^2})}{\Gamma(1 + \frac{\alpha+\bar{\alpha}}{b})}\, \mathrm{U}^{\rm NS}\left(\alpha, \bar{\alpha} + \frac{1}{b}\right),\end{aligned} \tag{4.7}$$

$$\begin{aligned}\mathcal{C}_2 \tilde{U}^{\rm NS}(-1/b, 0)\, \mathrm{U}^{R}(\alpha, \bar{\alpha}) &= \frac{\Gamma(\frac{\alpha+\bar{\alpha}}{b} - \frac{1}{b^2} + 1)}{(\frac{3}{2} - \alpha b)\Gamma(\frac{\alpha+\bar{\alpha}}{b} - \frac{2}{b^2})}\, \mathrm{U}^{R}\left(\alpha - \frac{1}{b}, \bar{\alpha}\right) - \\ &\quad - \tilde{\mu}' \frac{(\bar{\alpha} b + \frac{1}{2})\Gamma(\frac{\alpha+\bar{\alpha}}{b} - \frac{1}{b^2})}{\Gamma(1 + \frac{\alpha+\bar{\alpha}}{b})}\, \mathrm{U}^{R}\left(\alpha, \bar{\alpha} + \frac{1}{b}\right)\end{aligned} \tag{4.8}$$

with $\mathcal{C}_2 = \Gamma(1 - \frac{1}{b^2})/\Gamma(-\frac{2}{b^2})$. Here we have denoted one-point functions of the class-II degenerate field in terms of $\tilde{U}^{\rm NS}$ since they are in principle different from the one-point functions of general fields.

Notice that $\mathcal{C}_1$ contains $\Gamma(-1)$ in eq. (4.6) which arises from singular monodromy transformation. We can remove this singular factor by redefining $\mathrm{U}^{NS(R)}/\Gamma(-1) \to \mathrm{U}^{NS(R)}$. Notice that this redefinition does not change eqs. (4.7) and (4.8) since they are linear in $\mathrm{U}^{NS(R)}$ if assuming that $\tilde{U}^{\mathrm{NS}}$ is regular.

4.3 Solutions

The solutions to these equations can be expressed in terms of two integers $m, n \geq 1$ as follows:

$$\begin{aligned} \mathrm{U}^{\mathrm{NS}}_{mn}(\alpha,\bar\alpha) &= \mathcal{N}_{mn}(\pi\mu)^{-\frac{\alpha+\bar\alpha}{b}} \frac{\Gamma(\frac{\alpha+\bar\alpha}{b}-\frac{1}{b^2}+1)\Gamma(b(\alpha+\bar\alpha)-1)}{\Gamma(\alpha b)\Gamma(\bar\alpha b)} \times \\ &\times \sin\left[\frac{\pi m}{b}\left(\alpha+\bar\alpha-\frac{1}{b}\right)\right] \sin\left[\pi n b\left(\alpha+\bar\alpha-\frac{1}{b}\right)\right] \qquad (4.9) \\ \mathrm{U}^{R}_{mn}(\alpha,\bar\alpha) &= \mathcal{N}_{mn}(\pi\mu)^{-\frac{\alpha+\bar\alpha}{b}} \frac{\Gamma(\frac{\alpha+\bar\alpha}{b}-\frac{1}{b^2}+1)\Gamma(b(\alpha+\bar\alpha)-1)}{\Gamma(\alpha b-1/2)\Gamma(\bar\alpha b+1/2)} \times \\ &\times \sin\left[\frac{\pi m}{b}\left(\alpha+\bar\alpha-\frac{1}{b}\right)\right] \sin\left[\pi n b\left(\alpha+\bar\alpha-\frac{1}{b}\right)\right], \qquad (4.10) \end{aligned}$$

with the normalization factors given by

$$\mathcal{N}_{mn} = (-1)^n \frac{4b^2}{\Gamma(-1/b^2)} \frac{\cot(\pi n b^2)}{\sin(\pi m/b^2)}\,. \qquad (4.11)$$

This class of solutions will be associated with conformal BCs corresponding to the class-I neutral degenerate fields. It turns out that the conformal bootstrap equations do not allow discrete BCs corresponding to non-neutral degenerate fields. One possible explanation is that non-neutral BCs will introduce a boundary field which will not produce the identity operator when fused with bulk degenerate fields as they approach the boundary.

It is interesting to notice that the following one-point functions

$$\begin{aligned} \mathrm{U}^{\mathrm{NS}}_{m}(\alpha,\bar\alpha) &= \mathcal{N}_{m}(\pi\mu)^{-\frac{\alpha+\bar\alpha}{b}} \frac{\Gamma(1-\alpha b)\Gamma(1-\bar\alpha b)}{\Gamma(-\frac{\alpha+\bar\alpha}{b}+\frac{1}{b^2})\Gamma(2-b(\alpha+\bar\alpha))} \times \\ &\times \frac{\sin\left[\frac{\pi m}{b}(\alpha+\bar\alpha-\frac{1}{b})\right]}{\sin\left[\frac{\pi}{b}(\alpha+\bar\alpha-\frac{1}{b})\right]} \qquad (4.12) \\ \mathrm{U}^{R}_{m}(\alpha,\bar\alpha) &= \mathcal{N}_{m}(\pi\mu)^{-\frac{\alpha+\bar\alpha}{b}} \frac{\Gamma(\frac{3}{2}-\alpha b)\Gamma(\frac{1}{2}-\bar\alpha b)}{\Gamma(-\frac{\alpha+\bar\alpha}{b}+\frac{1}{b^2})\Gamma(2-b(\alpha+\bar\alpha))} \times \\ &\times \frac{\sin\left[\frac{\pi m}{b}(\alpha+\bar\alpha-\frac{1}{b})\right]}{\sin\left[\frac{\pi}{b}(\alpha+\bar\alpha-\frac{1}{b})\right]} \qquad (4.13) \\ \mathcal{N}_m &= \frac{\pi}{\Gamma(-\frac{1}{b^2}+1)} \frac{1}{\sin(\frac{\pi m}{b^2})} \qquad (4.14) \end{aligned}$$

satisfy eqs. (4.7) and (4.8). Although these do not satisfy eq. (4.5), hence not complete solutions, this class of solutions turns out to be consistent with modular bootstrap equations and we will show that they correspond to the class-III BCs.

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5. Modular bootstrap

In this section we derive the modular bootstrap equations based on the modular properties of degenerate characters. We derive the boundary amplitudes which are consistent with the one-point functions derived before.

5.1 Characters of general primary fields

The character of a CFT is defined by the trace over all the conformal states built on a specific primary state

$$\chi_h(q,y,t) = e^{2\pi ikt}\mathrm{Tr}\left[q^{L_0-c/24}y^{J_0}\right], \tag{5.1}$$

with $k = c/3$. Since the primary fields with general $\alpha, \bar{\alpha}$ of the $N = 2$ SLFT have no null states, the characters can be obtained by simply summing up all the descendant states. For these primary fields, it is more convenient to use the real parameters P, ω to denote them using eqs. (2.7) and (2.8). The (NS) character can be computed as

$$\chi^{\mathrm{NS}}_{[P,\omega]}(q,y,t) = e^{2\pi ikt}q^{P^2+b^2\omega^2/4}y^{\omega}\frac{\theta_{00}(q,y)}{\eta(q)^3}, \tag{5.2}$$

where we have introduced standard elliptic functions

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad \theta_{00}(q,y) = \prod_{n=1}^{\infty}\left[\left(1-q^n)(1+yq^{n-1/2}\right)\left(1+y^{-1}q^{n-1/2}\right)\right].$$

For the conformal BCs of super-CFTs, one needs to consider characters and associated Ishibashi states of the $(\widetilde{\mathrm{NS}})$ sectors [29]. The $(\widetilde{\mathrm{NS}})$ characters are defined by

$$\chi^{\widetilde{\mathrm{NS}}}_h(q,y,t) = e^{2\pi ikt}\mathrm{Tr}\left[(-1)^F q^{L_0-c/24}y^{J_0}\right]. \tag{5.3}$$

For a primary field $N_{\alpha\overline{\alpha}}$, $(-1)^F$ term contributes -1 for those descendants with odd number of $G^{\pm}_{-r}$. This can be efficiently incorporated into the character formula by shifting $y \to -y$ in the product. Therefore, the $(\widetilde{\mathrm{NS}})$ character is given by

$$\chi^{\widetilde{NS}}_{[P,\omega]}(q,y,t) = e^{2\pi ikt}q^{P^2+b^2\omega^2/4}y^{\omega}\frac{\theta_{00}(q,-y)}{\eta(q)^3}. \tag{5.4}$$

The character of a (R) primary field $R^{(\epsilon)}_{[P,\omega]}$ is given by

$$\chi^{R}_{[P,\omega,\epsilon]}(q,y,t) = e^{2\pi ikt}q^{P^2+b^2\omega^2/4}y^{\omega}\frac{\theta_{10}(q,y)}{\eta(q)^3}, \tag{5.5}$$

where we introduce another elliptic function

$$\theta_{10}(q,y) = (y^{1/2}+y^{-1/2})q^{1/8}\prod_{n=1}^{\infty}\left[(1-q^n)\left(1+yq^n)(1+y^{-1}q^n\right)\right]. \tag{5.6}$$

5.2 Characters of degenerate fields

In this subsection, we will consider only the (NS) fields which will be used later. The (R) characters can be similarly computed. Let us start with a class-I (NS) degenerate field $N^{\omega}_{m,n}$. As claimed in eq. (2.12), this field has a null state. Therefore, the character is given by

$$\chi^{\rm NS}_{mn\omega}(q,y,t) = e^{2\pi ikt}\left[q^{-\frac{1}{4}(\frac{m}{b}+nb)^2} - q^{-\frac{1}{4}(\frac{m}{b}-nb)^2}\right] q^{b^2\omega^2/4} y^{\omega}\frac{\theta_{00}(q,y)}{\eta(q)^3}\,. \tag{5.7}$$

The characters of the class-II (NS) degenerate fields, N^{ω}_{m} and $\tilde{N}^{\omega}_{m}$, are rather complicated due to the infinite null states structure. One should add and subtract contributions of these null states infinitely. The character of a class-IIA degenerate field is given by

$$\chi^{\rm NS}_{m\omega}(q,y,t) = e^{2\pi ikt}\frac{\theta_{00}(q,y)}{\eta(q)^3}\sum_{j=0}^{\infty}(-1)^j q^{-m^2/4b^2}(yq^{m/2})^{\omega+j} \tag{5.8}$$

$$= e^{2\pi ikt}\frac{y^{\omega}q^{-m^2/4b^2+m\omega/2}}{1+yq^{m/2}}\frac{\theta_{00}(q,y)}{\eta(q)^3} \tag{5.9}$$

and similarly for a class-IIB:

$$\chi^{\rm NS}_{m\omega}(q,y,t) = e^{2\pi ikt}\frac{y^{\omega}q^{-m^2/4b^2-m\omega/2}}{1+y^{-1}q^{m/2}}\frac{\theta_{00}(q,y)}{\eta(q)^3}\,. \tag{5.10}$$

One should be more careful for the neutral class-III degenerate fields. There are two infinite semi-chains of null states as expressed in eq. (2.23). Adding all these states, one can find the character as follows:

$$\chi^{\rm NS}_{m}(q,y,t) = e^{2\pi ikt}\frac{\theta_{00}(q,y)}{\eta(q)^3}\left[\sum_{j=0}^{\infty} q^{-m^2/4b^2}\left(-yq^{m/2}\right)^j + \sum_{j=1}^{\infty} q^{-m^2/4b^2}\left(-y^{-1}q^{m/2}\right)^j\right]$$

$$= e^{2\pi ikt}\frac{q^{-m^2/4b^2}(1-q^m)}{(1+yq^{m/2})(1+y^{-1}q^{m/2})}\frac{\theta_{00}(q,y)}{\eta(q)^3}. \tag{5.11}$$

When $m=1$, this character is the same as that of the identity operator as expected.

5.3 Modular transformations

The modular transformation of the class-I character can be easily found as

$$\chi^{\rm NS}_{mn\omega}(q',y',t') = 2b\int_{-\infty}^{\infty}dP\int_{-\infty}^{\infty}d\omega'\sinh\left(2\pi\frac{mP}{b}\right)\sinh(2\pi nbP)e^{-\pi ib^2\omega\omega'}\chi^{\rm NS}_{[P,\omega']}(q,y,t)\,. \tag{5.12}$$

Here we have used q', y', t' for the S-modular transformed parameters and ω' as the U(1) charge of a general primary field to distinguish it from that of the degenerate field ω.

The modular transformation of the class-II characters can be derived by the method of [30]:

$$\chi^{\rm NS}_{m\omega}(q',y',t') = \frac{b}{2}\int_{-\infty}^{\infty}dP\int_{-\infty}^{\infty}d\omega'\left[\frac{e^{-\pi i b^2\omega\omega'}\cosh[2\pi bP(\omega\mp 1\mp\frac{m}{b^2})]}{2\cosh(\pi bP+\frac{\pi i b^2\omega'}{2})\cosh(\pi bP-\frac{\pi i b^2\omega'}{2})}+\right.$$
$$\left.+\frac{e^{-\pi i b^2(\omega\mp 1)\omega'}\cosh[2\pi bP(\omega\mp\frac{m}{b^2})]}{2\cosh(\pi bP+\frac{\pi i b^2\omega'}{2})\cosh(\pi bP-\frac{\pi i b^2\omega'}{2})}\right]\times$$
$$\times\,\chi^{\rm NS}_{[P,\omega']}(q,y,t)\pm i\sum_{r\in\mathbb{Z}+\frac{1}{2}}\int_0^1 d\lambda e^{-i\pi(\pm m\lambda+2\omega r)}\tilde{\chi}^{\rm NS}_{r\lambda}(q,y,t)\,, \tag{5.13}$$

where the upper (lower) sign denotes the class-IIA (IIB), respectively.

For the class-III degenerate fields, the transformation is given by

$$\chi^{\rm NS}_{m}(q',y',t') = \frac{b}{2}\int_{-\infty}^{\infty}dP\int_{-\infty}^{\infty}d\omega'\frac{\sinh(2\pi mP/b)\sinh(2\pi bP)}{\cosh\left(\pi bP+\frac{\pi i b^2\omega'}{2}\right)\cosh\left(\pi bP-\frac{\pi i b^2\omega'}{2}\right)}\chi^{\rm NS}_{[P,\omega']}(q,y,t)+$$
$$+2\sum_{r\in\mathbb{Z}+\frac{1}{2}}\int_0^1 d\lambda\sin(\pi m\lambda)\tilde{\chi}^{\rm NS}_{r\lambda}(q,y,t). \tag{5.14}$$

Here we have defined a spectral flow of the class-IIA character

$$\tilde{\chi}^{\rm NS}_{r\lambda}(q,y,t)\equiv e^{2\pi ikt}\frac{y^{2r/b^2+\lambda}q^{r^2/b^2+r\lambda}}{1+yq^r}\frac{\theta_{00}(q,y)}{\eta(q)^3}. \tag{5.15}$$

5.4 Conformal boundary conditions

5.4.1 Vacuum BC

According to Cardy's formalism, one can associate a conformal BC with each primary state [14]. Among the conformal BCs of the $N=2$ SLFT, we concentrate on those associated with the degenerate fields. Following the modular bootstrap formulation, we can compute a boundary amplitude which is the inner product between the Ishibashi state of a primary state and the conformal boundary state. As usual, we start with the 'vacuum' BC amplitude [23, 22]:

$$\Psi^{\rm NS}_{\mathbf{0}}(P,\omega)\Psi^{\rm NS\dagger}_{\mathbf{0}}(P,\omega)=\mathbf{S}_{NS}(P,\omega)\,, \tag{5.16}$$

where the boundary amplitude is defined by

$$\Psi^{\rm NS}_{\mathbf{0}}(P,\omega)=\langle\mathbf{0}|N_{[P,\omega]}\rangle\rangle \tag{5.17}$$

and the modular S-matrix element $\mathbf{S}_{NS}(P,\omega)$ is given by eq. (5.14) with $m=1$.

Since $\Psi^{\rm NS\dagger}_{\mathbf{0}}(P,\omega)=\Psi^{\rm NS}_{\mathbf{0}}(-P,\omega)$, one can solve this up to some unknown constant as follows:

$$\Psi^{\rm NS}_{\mathbf{0}}(P,\omega)=\sqrt{\frac{b^3}{2}}(\pi\mu)^{-\frac{2iP}{b}}\frac{\Gamma\left(\frac{1}{2}-ibP+\frac{b^2\omega}{2}\right)\Gamma\left(\frac{1}{2}-ibP-\frac{b^2\omega}{2}\right)}{\Gamma\left(-\frac{2iP}{b}\right)\Gamma\left(1-2ibP\right)}. \tag{5.18}$$

Similarly, the vacuum boundary amplitude for the (R) Ishibashi state is given by [22]:

$$\Psi^{R}_{\mathbf{0}}(P,\omega)=-i\sqrt{\frac{b^3}{2}}(\pi\mu)^{-\frac{2iP}{b}}\frac{\Gamma\left(-ibP+\frac{b^2\omega}{2}\right)\Gamma\left(1-ibP-\frac{b^2\omega}{2}\right)}{\Gamma\left(-\frac{2iP}{b}\right)\Gamma\left(1-2ibP\right)}. \tag{5.19}$$

5.4.2 Class-I BCs

Now we impose the vacuum BC on one boundary and discrete BCs associated with the class-I degenerate fields on the other. Following the Cardy formalism, we can get the relation

$$\chi^{\rm NS}_{mn\omega}(q',y',t') = \int_{-\infty}^{\infty} dP \int_{-\infty}^{\infty} d\omega' \Psi^{\rm NS}_{mn\omega}(P,\omega')\Psi^{NS\dagger}_{\mathbf{0}}(P,\omega')\chi^{\rm NS}_{[P,\omega']}(q,y,t)\,. \tag{5.20}$$

Here, the boundary amplitudes are defined by an inner product between the the boundary state $|m,n,\omega\rangle$ and the Ishibashi state

$$\Psi^{\rm NS}_{mn\omega}(P,\omega) = \langle m,n,\omega|N_{[P,\omega]}\rangle\rangle\,. \tag{5.21}$$

Comparing this with eq. (5.12), we obtain

$$\Psi^{\rm NS}_{mn\omega}(P,\omega')\Psi^{NS\dagger}_{\mathbf{0}}(P,\omega') = 2b\sinh(2\pi mP/b)\sinh(2\pi nbP)e^{-\pi i b^2\omega\omega'}\,. \tag{5.22}$$

We can find the boundary amplitude from eq. (5.18)

$$\begin{aligned}\Psi^{\rm NS}_{mn\omega}(P,\omega') &= \sqrt{\frac{8}{b}}\,(\pi\mu)^{-\frac{2iP}{b}}\frac{\Gamma\left(\frac{2iP}{b}\right)\Gamma\left(1+2ibP\right)}{\Gamma\left(\frac{1}{2}+ibP+\frac{b^2\omega}{2}\right)\Gamma\left(\frac{1}{2}+ibP-\frac{b^2\omega}{2}\right)}\times\\ &\quad\times\sinh\left(2\pi\frac{mP}{b}\right)\sinh(2\pi nbP)e^{-\pi i b^2\omega\omega'}\,.\end{aligned} \tag{5.23}$$

It is remarkable that this solution coincides with eq. (4.9), the ZZ-brane solution with BC $(m,n,\omega=0)$. This provides a most important consistency check between the conformal and modular bootstraps.

Now we impose the class-I discrete BCs on both boundaries. The partition function is expressed by

$$Z^{\rm NS}_{(mn\omega)(m'n'\omega')}(q',y',t') = \int_{-\infty}^{\infty} dP \int_{-\infty}^{\infty} d\omega'' \chi^{\rm NS}_{[P,\omega'']}(q,y,t)\Psi^{\rm NS}_{mn\omega}(P,\omega'')\Psi^{NS\dagger}_{m'n'\omega'}(P,\omega'')\,. \tag{5.24}$$

Inserting eq. (5.23) into this, we find

$$Z^{\rm NS}_{(mn\omega)(m'n'\omega')} = \sum_{k=|m-m'|+1}^{m+m'-1}\ \sum_{l=|n-n'|+1}^{n+n'-1}\left[\chi^{\rm NS}_{k,l-1,\omega+\omega'}+\chi^{\rm NS}_{k,l+1,\omega+\omega'}+\chi^{\rm NS}_{k,l,\omega+\omega'+1}+\chi^{\rm NS}_{k,l,\omega+\omega'-1}\right], \tag{5.25}$$

where we have omitted the modular parameters for simplicity. From eq. (5.25) we can read off the fusion rules of the class-I degenerate fields. In particular, the OPE between neutral fields are given by

$$N^0_{mn}N^0_{m'n'} = \sum_{k=|m-m'|+1}^{m+m'-1}\ \sum_{l=|n-n'|+1}^{n+n'-1}\left[N^0_{k,l-1}+N^0_{k,l+1}+N^1_{k,l}+N^{-1}_{k,l-1}\right]. \tag{5.26}$$

Notice that this fusion rule can not be applicable to $n=n'$ case where $l=1,\ldots,2n-1$. If $l=1$, the field $N^1_{k,1}$ is not in class-I as mentioned before and the relation (5.25) is not valid. This explains why the OPE of two identical class-I degenerate fields includes identity field which is not in class-I but in class-III.

So far we have considered the discrete BCs on both boundaries. It is interesting to consider a mixed BC, namely a class-I BC on one boundary and the continuous (FZZT) BC [22] on the other. In this case, the partition function can be written as

$$Z^{\rm NS}_{(mn\omega)s}(q',y',t') = \int_{-\infty}^{\infty} dP \int_{-\infty}^{\infty} d\omega' \chi^{\rm NS}_{[P,\omega']}(q,y,t)\Psi^{\rm NS}_{mn\omega}(P,\omega')\Psi_s^{NS\dagger}(P,\omega')\,, \qquad (5.27)$$

with

$$\Psi^{\rm NS}_{\mathbf{s}}(P,\omega') = \sqrt{2b^3}\,(\pi\mu)^{-\frac{2iP}{b}} \frac{\Gamma\left(1+\frac{2iP}{b}\right)\Gamma\left(2ibP\right)\cos(4\pi sP)}{\Gamma\left(\frac{1}{2}+ibP+\frac{b^2\omega'}{2}\right)\Gamma\left(\frac{1}{2}+ibP-\frac{b^2\omega'}{2}\right)}, \qquad (5.28)$$

which is in fact eq. (3.39) up to a proportional constant. Inserting eq. (5.23) into this, we obtain the following result:

$$Z^{\rm NS}_{(mn\omega)s} = \sum_{k=1-m,2}^{m-1}\sum_{l=1-n,2}^{n-1}\left[\chi^{\rm NS}_{[P_{-k,-l-1},\omega]} + \chi^{\rm NS}_{[P_{k,l+1},\omega]} + \chi^{\rm NS}_{[P_{k,l},\omega+1]} + \chi^{\rm NS}_{[P_{k,l},\omega-1]}\right], \qquad (5.29)$$

where we have omitted the modular parameters for simplicity and introduced a momentum variable

$$P_{k,l} = s + \frac{i}{2}\left(\frac{k}{b}+lb\right). \qquad (5.30)$$

It is more illustrative to consider the most simple case, namely, $m = n = 1, \omega = 0$. This BC is associated with the class-I degenerate field $N_{-b/2}$ which we have considered in section 3.2. The above equation is simplified to

$$Z^{\rm NS}_{(110)s} = \chi^{\rm NS}_{[s-ib/2,0]} + \chi^{\rm NS}_{[s+ib/2,0]} + \chi^{\rm NS}_{[s,1]} + \chi^{\rm NS}_{[s,-1]}\,. \qquad (5.31)$$

In terms of $\alpha, \bar{\alpha}$, one can easily check that these are characters of the operators appearing in the OPE (3.4). This provides another consistency check for our results.

5.4.3 Class-II BCs

For the class-II and class-III (neutral) BCs, there are two types of Ishibashi states flowing in the bulk. One is associated with the continuous state $|N_{[P,\omega']}\rangle\rangle$ and the other with class-II degenerate fields and their spectral flows. We denote this Ishibashi state by $|r,\lambda\rangle\rangle$. The appearance of this state can be understood from the modular transformations, eq. (5.13).

If we denote the class-II boundary state $|m,\omega\rangle$, we can define the following boundary amplitudes as inner products between the boundary state and the Ishibashi states

$$\Psi^{\rm NS}_{m\omega}(P,\omega') = \langle m,\omega|N_{[P,\omega']}\rangle\rangle\,, \qquad \Phi^{\rm NS}_{m\omega}(r,\lambda) = \langle m,\omega|r,\lambda\rangle\rangle\,. \qquad (5.32)$$

Using these, one can express the partition function with the vacuum BC on one side and a class-II BC on the other boundary

$$\begin{aligned}\chi^{\rm NS}_{m\omega}(q',y',t') = & \int_{-\infty}^{\infty} dP\int_{-\infty}^{\infty} d\omega' \Psi^{\rm NS}_{m\omega}(P,\omega')\Psi_{\mathbf{0}}^{NS\dagger}(P,\omega')\chi^{\rm NS}_{[P,\omega']}(q,y,t) + \\ & + \sum_{r\in Z+\frac{1}{2}}\int_0^1 d\lambda \Phi^{\rm NS}_{m\omega}(r,\lambda)\Phi_1^{NS\dagger}(r,\lambda)\tilde{\chi}^{\rm NS}_{r\lambda}(q,y,t)\,. \end{aligned} \qquad (5.33)$$

Comparing this with eq. (5.13), we obtain

$$\Psi^{\rm NS}_{m\omega}(P,\omega')\Psi^{NS\dagger}_{\mathbf{0}}(P,\omega') = S_{m\omega}(P,\omega')\,, \tag{5.34}$$

where $S_{m\omega}(P,\omega')$ is the modular S-matrix component in eq. (5.13). From this, one can solve for $\Psi^{\rm NS}_{m\omega}(P,\omega')$. Instead of presenting details for this case, we will analyze more interesting case, namely the neutral ($\omega = 0$) class-III BCs.

5.4.4 Class-III BCs

For a class-III (neutral) boundary state $|m\rangle$, we can define two boundary amplitudes

$$\Psi^{\rm NS}_{m}(P,\omega) = \langle m|N_{[P,\omega]}\rangle\rangle, \qquad \Phi^{\rm NS}_{m}(r,\lambda) = \langle m|r,\lambda\rangle\rangle \tag{5.35}$$

due to the two Ishibashi states. Imposing this BC on one side and the vacuum BC on the other, we can find

$$\begin{aligned}\chi^{\rm NS}_{m}(q',y',t') &= \int_{-\infty}^{\infty} dP \int_{-\infty}^{\infty} d\omega' \Psi^{\rm NS}_{m}(P,\omega')\Psi^{NS\dagger}_{\mathbf{0}}(P,\omega') + \\ &+ \sum_{r\in\mathbb{Z}+\frac{1}{2}} \int_0^1 d\lambda \Phi^{\rm NS}_{m}(r,\lambda)\Phi^{NS\dagger}_{1}(r,\lambda)\tilde{\chi}^{\rm NS}_{r\lambda}(q,y,t)\,.\end{aligned} \tag{5.36}$$

Comparing with eq. (5.14), we obtain

$$\Psi^{\rm NS}_{m}(P,\omega) = \Psi^{\rm NS}_{\mathbf{0}}(P,\omega)\frac{\sinh(2\pi mP/b)}{\sinh(2\pi P/b)}\,, \tag{5.37}$$

and

$$\Phi^{\rm NS}_{m}(r,\lambda) = \frac{2\sin(m\pi\lambda)}{\sqrt{2\sin(\pi\lambda)}}\,. \tag{5.38}$$

The solution (5.37) coincides with the one-point function (4.12). Imposing these BCs on both boundaries, the partition function is given by

$$\begin{aligned}Z^{\rm NS}_{mm'}(q',y',t') &= \int_{-\infty}^{\infty} dP \int_{-\infty}^{\infty} d\omega \chi^{\rm NS}_{[P,\omega]}(q,y,t)\Psi^{\rm NS}_{m}(P,\omega)\Psi^{NS\dagger}_{m'}(P,\omega) + \\ &+ \sum_{r\in\mathbb{Z}+\frac{1}{2}} \int_0^1 d\lambda \Phi^{\rm NS}_{m}(r,\lambda)\Phi^{NS\dagger}_{m'}(r,\lambda)\tilde{\chi}^{\rm NS}_{r\lambda}(q,y,t)\,.\end{aligned} \tag{5.39}$$

Inserting eqs. (5.37) and (5.38) into this, one can express it as

$$Z^{\rm NS}_{mm'}(q',y',t') = \sum_{k=|m-m'|+1}^{m+m'-1} \chi^{\rm NS}_{k}(q',y',t')\,. \tag{5.40}$$

This is a desired fusion rule of the neutral degenerate fields.

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5.4.5 Modular bootstrap for the (R) sector

One can perform similar analysis for the (R) sector. The ($\widetilde{\mathrm{NS}}$) characters are related to the (R) characters by the following relation

$$\chi^{\widetilde{NS}}_{mn\omega}(q',y',t') = \int_{-\infty}^{\infty} dP \int_{-\infty}^{\infty} d\omega' \Psi^{R}_{mn\omega}(P,\omega')\Psi^{R\dagger}_{\mathbf{0}}(P,\omega')\chi^{R}_{[P,\omega']}(q,y,t)\,. \tag{5.41}$$

Comparing with the modular S-matrix element, we can find

$$\Psi^{R}_{mn\omega}(P,\omega')\Psi^{R\dagger}_{\mathbf{0}}(P,\omega') = 2b\sinh\left(2\pi\frac{mP}{b}\right)\sinh(2\pi nbP)e^{-\pi i b^2\omega\omega'} \tag{5.42}$$

from which we can find

$$\begin{aligned}\Psi^{R}_{mn\omega}(P,\omega') &= -i\sqrt{\frac{8}{b}}\,(\pi\mu)^{-\frac{2iP}{b}}\frac{\Gamma\left(\frac{2iP}{b}\right)\Gamma\left(1+2ibP\right)}{\Gamma\left(ibP+\frac{b^2\omega}{2}\right)\Gamma\left(1+ibP-\frac{b^2\omega}{2}\right)}\times\\ &\quad\times\sinh\left(2\pi\frac{mP}{b}\right)\sinh(2\pi nbP)e^{-\pi i b^2\omega\omega'}\,.\end{aligned} \tag{5.43}$$

It is straightforward to continue this analysis for the class-II and class-III BCs and their mixed BCs for the (R) sector.

6. Discussions

In this paper we have derived conformal bootstrap equations for the $N = 2$ SLFT on a half plane with appropriate boundary action and on a pseudosphere. We have found the solutions of these functional equations which correspond to conformal BCs. We have also checked the consistency of these solutions by the modular bootstrap analysis. In particular, we have found a new class of 'discrete' conformal BCs of the $N = 2$ SLFT which are parameterized by two positive integers and U(1) charge. These solutions are associated with class-I degenerate fields. When U(1) charge vanishes, it is tempting to interpret these solutions as D0-branes in 2D fermionic black hole. The solutions with generic integer values may describe non-BPS, hence, unstable D0-brane. An interesting case arises when $n = m = 1$. As we mentioned, this is different from the vacuum BC. Our solution seems to suggest new boundary state for the 2D string theories.

Another intriguing point is the resemblance of the class-III solutions with D0-brane solutions of the $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset CFT [24] which is dual to the sine-Liouville theory [4, 5]. Since the $N = 2$ SLFT is dual to the fermionic $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset CFT [6, 7], it is natural that the two coset theories are closely related.

This relation between the coset theories means that the $N = 2$ SLFT is closely related to the sine-Liouville theory. This can be checked by comparing the bulk reflection amplitudes. We expect that this relationship still exists in the presence of boundary. It is an interesting open problem to derive one-point functions based on the conformal bootstrap of the sine-Liouville theory and compare with the results obtained in this paper.

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